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Refining the Hierarchy of Blind Multicounter Languages

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Abstract

We show that the families (k,r)–RBC of languages accepted (in quasi-realtime) by one-way counter automata having k blind counters of which r are reversal-bounded form a strict and linear hierarchy of semi-AFLs. This hierarchy comprises the families $BLIND = \mathcal{M}_\cap(C_1)$ of blind multicounter languages with generator $C_1 := \{w \in \{a_1, b_1\}^* \mid |w|_{a_1} = |w|_{b_1}\}$ and $RBC = \mathcal{M}_\cap(B_1)$ of reversal-bounded multicounter languages with generator $B_1 := \{a_1^n b_1^n \mid n \in I\!N\}$. This generalizes and sharpens the known results from [Grei 78] and [Jant 98]. The proof for the strict inclusions for the first time uses techniques from linear algebra.

Zusammenfassung

Wir zeigen, dass die Familien (k,r)–RBC der Sprachen, die in Quasi-Realzeit von Zählerautomaten mit k blinden und r umkehrbeschränkten Zählern akzeptiert werden, eine echte und lineare Hierarchie von Sprachfamilien bildet. Diese Hierarchie enthält sowohl die bekannte Sprachfamilie $BLIND = \mathcal{M}_\cap(C_1)$ der blinden Mehrzählersprachen mit Generatormenge $C_1 := \{w \in \{a_1, b_1\}^* \mid |w|_{a_1} = |w|_{b_1}\}$, als auch die Familie $RBC = \mathcal{M}_\cap(B_1)$ der umkehrbeschränkten Mehrzählersprachen. Mit diesem Resultat werden die bekannten Ergebnisse von Greibach, [Grei 78], und Jantzen, [Jant 98], verschärft und verallgemeinert. Die hier benutzte Methode verwendet erstmalig Resultate der linearen Algebra.

1 Introduction

Hierarchies of counter automata are often proved by arguments concerning the dimension of the memory space, i.e., the number of counters, see for example [FiMR 68, Grei 76, Grei 78] or counting cycles within the computations, as in [Hrom 86]. If one does not alter the dimension, and changes only the strategy of accessing the counters, other methods have to be found. The method applied here for the first time uses techniques from linear algebra and shows that the formerly known two hierarchies of blind and of reversal-bounded multicounter languages are in fact part of one linear hierarchy of semi-AFLs.

The family of languages accepted by one-way reversal-bounded multicounter automata is a well known semi-AFL which is principal as an *intersection*-closed semi-AFL $\mathcal{M}_\cap(B_1)$ with generator $B_1 := \{a_1^n b_1^n \mid n \in I\!N\}$, which is not a principal semi-AFL, see [FiMR 68, Grei 78]. It was shown in [Grei 78] that arbitrary blind multicounter automata accept exactly the same languages than those working in quasi real-time.

The known situation for these hierarchies, shown in [Grei 78], is as follows: $\bigcup_{i\geq 1}\mathcal{M}(C_i) = \mathcal{M}_\cap(C_1) = \mathit{BLIND} = \bigcup_{i\geq 1}\mathcal{M}(B_i) = \mathcal{M}_\cap(B_1) = \mathit{RBC}$, where $\mathcal{M}(\mathcal{L})$ denotes the least trio generated by the family $\mathcal{L}$, which is a semi-AFL if $\mathcal{L} = \{L\}$ and then we write $\mathcal{M}(L)$ instead of $\mathcal{M}(\mathcal{L})$. For all $i \geq 1$ we have $\mathcal{M}(B_i) \subsetneq \mathcal{M}(B_{i+1})$, see [Gins 75], and $\mathcal{M}(C_i) \subsetneq \mathcal{M}(C_{i+1})$, shown in [Grei 76, Grei 78]. (For the definition of the languages B_i and C_i see Definition 2.1 below).

We study the families (k,r)–RBC of languages accepted (in quasi-realtime) by one-way (or on-line) counter automata having k blind counters of which $r \leq k$ are reversal-bounded and prove (k_1,r_1)–$\mathit{RBC} \subsetneq (k_2,r_2)$–$\mathit{RBC}$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $r_1 > r_2$. Then $(k,0)$–$\mathit{RBC} = \mathcal{M}(C_k)$, and $\bigcup_{k\geq 1}(k,0)$–$\mathit{RBC} = \bigcup_{i\geq 1}\mathcal{M}(C_i) = \mathcal{M}_\cap(C_1)$ forms a hierarchy of *twist*-closed semi-AFLs (see [Jant 98]). The strict inclusions are proved here for the first time and the restriction to quasi-realtime is neither used nor needed.

2 Basic Definitions

2.1 Definition

For any alphabet Σ and $x \in \Sigma$ let $|w|_x$ denote the number of occurrences of the symbol x within the string $w \in \Sigma^*$, $|w| := \sum_{x\in\Sigma} |w|_x$, and $\psi : \Sigma^* \longrightarrow I\!N^n$ is the Parikh mapping, defined by $\psi(w) := (|w|_{x_1}, \ldots, |w|_{x_n})$, where $n := |\Sigma|$. As usual, this (and similarly other mappings) defined on strings are continued to sets L of strings by $\psi(L) := \{\psi(w) \mid w \in L\}$. The empty word will be denoted by λ and $\psi(\lambda) = \vec{0} := (\underbrace{0, 0, \ldots, 0}_{\text{n-times}}) \in I\!N^n$ where $I\!N := \{0, 1, 2, \ldots\}$ denots the set of non-negative integers.

The languages we use here are constructed using the specific alphabet Γ_n specified for each $n \in I\!N, n \geq 1$ by: $\Gamma_n := \{a_i, b_i \mid 1 \leq i \leq n\}$, and the homomorphisms h_i defined for $i \geq 1$ by: $h_i(x) := \begin{cases} x, & \text{if } x \in \{a_i, b_i\} \\ \lambda, & \text{else} \end{cases}$.

$$
\begin{aligned}
C_n &:= \left\{ w \in \Gamma_n^* \mid \forall\, 1 \leq i \leq n : |w|_{a_i} = |w|_{b_i} \right\} \\
B_n &:= \left\{ w \in C_n^* \mid \forall\, 1 \leq i \leq n : h_i(w) = a_i^m b_i^m, \text{ for some } m \in I\!N \right\} \\
D_n &:= \left\{ w \in \Gamma_n^* \mid \forall\, 1 \leq i \leq n : \left(|w|_{a_i} = |w|_{b_i} \wedge \forall w = uv : |u|_{a_i} \geq |u|_{b_i} \right) \right\}
\end{aligned}
$$

□

The language D_1 defined above is the so-called semi-Dyck language on one pair of brackets which is often abbreviated by $D_1'^*$, see e.g. [Bers 80]. D_n here denotes the n-fold shuffle of disjoint copies of the semi-Dyck language D_1 and it is known, [Grei 78, Jant 79], that $\bigcup_{i\geq 1} \mathcal{M}(D_i) = \mathcal{M}_\cap(D_1) = PBLIND(n)$. The latter family consists of languages accepted in quasi-realtime by nondeterministic one-way multicounter acceptors which operate in such a way that in every computation no counter can store a negative value, and the information on whether or not the value stored in a counter is *zero* is not used for deciding the next move.

The languages C_n are the (symmetric) Dyck languages on n pairs of brackets a_i, b_i, often abbreviated by D_n^*, see again [Bers 80]. Greibach, [Grei 78], has shown that

$$
\bigcup_{i\geq 1} \mathcal{M}(C_i) = \mathcal{M}_\cap(C_1) = BLIND = BLIND(lin) = BLIND(n) =
$$

$$
\bigcup_{i\geq 1} \mathcal{M}(B_i) = \mathcal{M}_\cap(B_1) = RBC(n) = RBC \subsetneq PBLIND
$$

.

Here *BLIND* (*BLIND*(n), *BLIND*(lin)) denotes the family of languages accepted (in quasi-realtime, linear time, resp.) by nondeterministic one-way multicounter acceptors which operate in such a way that in every computation all counters may store arbitrary integers, and the information on the contents of the counters is not used for deciding the next move. The family RBC is the family of languages accepted by nondeterministic one-way multicounter acceptors performing at most one reversal in each computation. The formal definition is to be found in Section 3.

3 Blind k counter automata with $r \leq k$ reversal-bounded counters.

We shall deal only with counter-automata that have a one-way read-only input tape (also known as on-line automata) and have k-blind counters of which precisely r counters are reversal-bounded.

3.1 Definition

A blind k-counter automaton $M := (Q, \Sigma, \delta, q_0, Q_{fin})$ consists of a finite set of states Q, a designated initial state $q_0 \in Q$, a designated set of final states $Q_{fin} \subseteq Q$, a finite input alphabet Σ, and a transition function $\delta : Q \times (\Sigma \cup \{\lambda\}) \to 2^{Q \times \{+1,0,-1\}^k}$.

An instantaneous description (ID) of M is an element of $Q \times \Sigma^* \times \mathbb{Z}^k$. We write $(q_1, aw, z_1, \ldots, z_k) \vdash_M (q_2, w, z_1 + \Delta(1), \ldots, z_k + \Delta(k))$ if $(q_2, \Delta) \in \delta(q_1, a)$, where $\Delta' = (\Delta(1), \ldots, \Delta(k))$ is the transpose of column vector Δ. $\vdash^*_M$ denotes the reflexive transitive closure of the computation relation $\vdash_M$ and is defined as usual from the n-step computation relations $\vdash^n_M := \vdash^{n-1}_M \circ \vdash_M$ by $\vdash^*_M := \bigcup_{i \geq 0} \vdash^i_M$, where $\vdash^0_M$ is the identity relation on the ID's of the nondeterministic automaton M and we omit the subscript M if no confusion will arise.

$ID_i \vdash^*_M ID_j$ is an accepting computation for w iff $ID_i := (q_0, w, 0, \ldots, 0))$ and $\exists q_e \in Q_{fin}$ such that $ID_j := (q_e, \lambda, 0, \ldots, 0))$.

$L(M) := \{w \in \Sigma^* \mid M$ has an accepting computation for $w\}$ is the language accepted by M.

A specific k-counter automaton M can most easily be described by a finite state transition diagram in which a directed arc from state q_1 to q_2 is inscribed by the input symbol x to be processed and a vector $\Delta \in \{+1, 0, -1\}^k$ used for updating the counters by adding the component $\Delta(i)$ of Δ to the current contents z_i of the i-th counter. This will be written as $q_1 \xrightarrow[\Delta]{x} q_2$.

□

3.2 Definition

A blind k-counter automaton $M := (Q, \Sigma, \delta_M, q_0, Q_{fin})$ accepts $L(M)$ in in linear time with factor $d \in \mathbb{N}$, if for any $w \in L(M)$ there exists an accepting n-step computation $ID_0 \vdash^n_M ID_1$ for w such that $n \leq d \cdot max(|w|, 1)$.

If there exists $d \in I\!N$ such that $(q_1, \lambda, z_1, \ldots, z_k) \vdash_M^n (q_2, \lambda, z_1', \ldots, z_k')$ implies $n \leq d$, then the automaton M is said to work in quasi-realtime of delay d. If in this case $d = 0$ then M works in realtime.

The i-th counter $(1 \leq i \leq k)$ of some blind k-counter automaton M is reversal-bounded iff for any subcomputation
$(q_0, w, 0, \ldots, 0) \vdash_M^* (q_1, w_1, x_1, \ldots, x_k) \vdash_M^* (q_2, w_2, y_1, \ldots, y_k) \vdash_M^* (q_3, w_3, z_1, \ldots, z_k)$ $x_i > y_i$ implies $y_i \geq z_i$. □

By this definition, a reversal-bounded counter has to be increased first and decreased after its reversal. Counters that are first decreased and solely increased after one reversal can be replaced by those required by Definition 3.2 above. In addition, reversal bounded counters are forced by the finite control to perform at most one reversal on each computation, even in the non-accepting ones!

3.3 Definition

For all $k, r \in I\!N$ let (k, r)–RBC denote the family of languages accepted by (k, r)-counter automata, i.e., are accepted by on-line counter automata having k blind counters of which r are reversal-bounded. □

Obviously we have $\mathcal{M}(C_k) = (k, 0)$–$RBC$ and $\mathcal{M}(B_k) = (k, k)$–$RBC$.

3.4 Definition

$L_{k,r} := \{w \in \Gamma_k^* \mid \forall_{1 \leq i \leq r} : h_i(w) = a_i^m b_i^m \text{ for some } m \in I\!N \wedge \forall_{r+1 \leq i \leq k} : |w|_{a_i} = |w|_{b_i}\}$. □

By results from Ginsburg and Greibach ([GiGr 70], Corr. 3, and [Gins 75], Prop. 3.6.1,) one can deduce that the language $L_{k,r}$ is a generator of the family (k, r)–RBC. We do not give a detailed explanation using these standard techniques and state Lemma 3.5 without proof:

3.5 Lemma

(k, r)–$RBC = \mathcal{M}(L_{k,r})$.

Greibach showed $C_1 \in \mathcal{M}(B_3)$, (Lemma 1 in [Grei 78]). It was shown in [Jant 98] that it is sufficient to accept C_k using only k+1 reversal-bounded counters, which is stated in Lemma 3.6.

3.6 Lemma

$\forall k \in I\!N, k \geq 1 : \mathcal{M}(C_k) \subsetneq \mathcal{M}(B_{k+1})$.

Ginsburg ([Gins 75] Example 4.5.2) has shown $\mathcal{M}(B_k) \subsetneq \mathcal{M}(B_{k+1})$. And $\mathcal{M}(C_i) \subsetneq \mathcal{M}(C_{i+1})$ has been shown in [Grei 76], [Grei 78].

We will obtain the sharpening of the above results by proving Lemma 3.14 and the main result Theorem 3.22.

For the formulation and usage of techniques from linear algebra to prove these results we need some more notation that in most cases applies only to those (k, r)-counter automata which accept languages from Γ_k^*.

3.7 Definition

For any (k, r)-counter automaton $A := (Q, \Sigma, \delta_A, q_0, Q_{fin})$ let $G_A \subseteq Q \times (\Sigma \cup \{\lambda\}) \times \{+1, 0, -1\}^k \times Q$ be the finite set defined by $G_A := \{(p, x, \Delta, q) \mid (q, \Delta) \in \delta_A(p, x)\}$, which is in bijection with the arcs of A's state diagram. For later use let $n_A := |G_A|$ be the number of elements in the arbitrarily but fixed ordered set $G_A = \{g_1, g_2, \ldots, g_{n_A}\}$. (The ordering that is actually used depends on $L(A)$ and will be described later.)

The four projections $\pi_i, 1 \leq i \leq 4$, $\pi_1, \pi_4 : G_A \to Q$, $\pi_2 : G_A \to \Sigma \cup \{\lambda\}$, and $\pi_3 : G_A \to \{+1, 0, -1\}^k$, are defined by: $\pi_1((p, x, \Delta, q)) := p$, $\pi_2((p, x, \Delta, q)) := x$, $\pi_3((p, x, \Delta, q)) := \Delta$, $\pi_4((p, x, \Delta, q)) := q$.

The mappings π_1 and π_4 are mere coding, whereas π_2 and π_3 are canonically extended to homomorphisms, by mild abuse of notation: For all strings $u, v \in G_A^*$ let $\pi_2 : G_A^* \to \Sigma^*$ with $\pi_2(uv) = \pi_2(u)\pi_2(v)$ and $\pi_3 : G_A^* \to \mathbb{Z}^k$ with $\pi_3(uv) = \pi_3(u) + \pi_3(v)$, where $+$ is the componentwise addition of the vectors $\pi_3(u)$ and $\pi_3(v)$. For an easier readability let $\Delta_g := \pi_3(g)$ denote the counter update induced by the transition $g \in G_A$ of A.

Let $R_A := \{g_{i_0} g_{i_1} \cdots g_{i_t} \mid t \in I\!N \wedge \forall \mu \in \{0, \ldots t\} : (g_{i_\mu} \in G_A) \wedge (\pi_1(g_{i_0}) = q_0) \wedge (\pi_4(g_{i_t}) \in Q_{fin}) \wedge (\pi_4(g_{i_\mu}) = \pi_1(g_{i_{\mu+1}})$ for $\mu \neq t)\} \subseteq G_A^*$ be the regular set describing all the accepting paths in A's state diagram, interpreted as finite automaton with input alphabet G. □

Of course, $w \in R_A$ does not imply that $\pi_2(w)$ will be accepted by the counter automaton A, since the final counter values may not be equal to *zero*. Note, that the number of reversals of the reversal-bounded counters are handled by the finite control and can never be wrong.

On the basis of a (k, r)-counter automaton $A := (Q, \Gamma_k, \delta_A, q_0, Q_{fin})$ two matrices A_Δ and A_Γ are defined.

3.8 Definition

$A_\Delta \in \mathbb{Z}^{k \times n_A}$ is defined for each component, $1 \leq i \leq k$, $1 \leq j \leq n_A$ by:

$$A_\Delta(i, j) := \Delta_{g_j}(i).$$

$\square$

Hence A_Δ can be written as composite matrix as follows:

$$A_\Delta = \left(\Delta_{g_1}\, \Delta_{g_2}\, \cdots\, \Delta_{g_{n_A}} \right)$$

With the notation from Definition 3.7 we see that $A_\Delta \cdot \psi(v) = \pi_3(v)$ for each $v \in G_A^*$ and the following is a consequence of the definition of acceptance for (k,r)-counter automata:

3.9 Lemma

Let $A := (Q, \Sigma, \delta_A, q_0, Q_{fin})$ be some (k,r)-counter automaton then

$$\forall\, v \in R_A : \quad A_\Delta \cdot \psi(v) = \vec{0} \quad \text{iff} \quad \pi_2(v) \in L(A).$$

Proof: $v \in R_A$ ensures that there exists a path in the state diagram of A beginning in q_0 and ending in some final state of Q_{fin}. If in addition $\pi_3(v) = A_\Delta \cdot \psi(v) = 0$, then $\pi_2(v) \in L(A)$. Conversely, for any $w \in L(A)$ there exists an accepting path in A having a corresponding string $v' \in R_A$ with $w = \pi_2(v')$. Since a (k,r)-counter automaton accepts if the k-counters are empty at the beginning and at the end, it follows that $\pi_3(v') = A_\Delta \cdot \psi(v') = \vec{0}$. $\square$

3.10 Definition

For each (k,r)-counter automaton $A := (Q, \Gamma_k, \delta_A, q_0, Q_{fin})$ the following matrix $A_\Gamma \in \{+1, 0, -1\}^{k \times n_A}$ is defined for each component $A_\Gamma(i,j)$, $1 \le i \le k$, $1 \le j \le n_A$, by:

$$A_\Gamma(i,j) := \begin{cases} 1 & \text{, if } \pi_2(g_j) = a_i \\ -1 & \text{, if } \pi_2(g_j) = b_i \\ 0 & \text{, if } \pi_2(g_j) \notin \{a_i, b_i\}. \end{cases}$$

$\square$

Without loss of generality the ordering of the elements in G_A is such, that

$$A_\Gamma = \begin{pmatrix} 1\cdots 1 & -1\cdots -1 & 0\cdots 0 & 0\cdots 0 & \cdots & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ 0\cdots 0 & 0\cdots 0 & 1\cdots 1 & -1\cdots -1 & \cdots & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \multicolumn{8}{c}{\dotfill} \\ 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & \cdots & 1\cdots 1 & -1\cdots -1 & 0\cdots 0 \end{pmatrix}$$

$$= \left(\gamma_1\, \gamma_2\, \cdots\, \gamma_{n_A} \right),$$

where γ_j denotes the $j-$th column $A_\Gamma(:,j)$ of A_Γ.

The next fact is obvious from the definitions and formulated without proof:

3.11 Lemma

Let $A := (Q, \Gamma_k, \delta_A, q_0, Q_{fin})$ be some (k, r)-counter automaton then

$$\forall\, v \in G_A^* : \quad A_\Gamma \cdot \psi(v) = \vec{0} \quad \text{iff} \quad \pi_2(v) \in C_k.$$

We combine the preceding Lemmas (3.9 and 3.11) to get an equality which is independent from the number of reversal bounded counters but, through R_A, not independent of the language accepted:

3.12 Lemma

Let $A := (Q, \Gamma_k, \delta_A, q_0, Q_{fin})$ be some (k, r)-counter automaton accepting $L(A) \subseteq C_k$, and let $\begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix}$ denote the compound matrix of dimension $2k \times n_A$ then

$$\{v \in R_A \mid A_\Delta \cdot \psi(v) = \vec{0}\} = \{v \in R_A \mid \begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot \psi(v) = \vec{0}\}.$$

3.13 Definition

$$B_{k,r} := \{a_1^{i_1} b_1^{i_1} \cdots a_r^{i_r} b_r^{i_r} a_{r+1}^{i_{r+1}} b_{r+1}^{i_{r+1}+j_{r+1}} a_{r+1}^{j_{r+1}} \cdots a_k^{i_k} b_k^{i_k+j_k} a_k^{j_k} \mid \forall\, \mu : i_\mu, j_\mu \in I\!N\}.$$

□

3.14 Lemma

$(k, r+1)$–$RBC \neq (k, r)$–RBC for all $k \in I\!N$ and $0 \leq r < k$.

We will in fact prove $B_{k,r} \notin (k, r+1)$–RBC, where the subset $B_{k,r} \subsetneq L_{k,r}$ has defined above (Def. 3.13). We will see, that the equation of Lemma 3.12 cannot be satisfied if $B_{k,r}$ is accepted by using r+1 reversal bounded counters. That this suffices is obvious, since $B_{k,r}$ is obtained from $L_{k,r}$ by intersection with an appropriate bounded regular set, hence $B_{k,r} \in (k, r)$–RBC is easily seen.

The proof of Lemma 3.14 is quite involved and needs a lot of definitions first. For the sake of contradiction, let us assume $B_{k,r} \in (k, r{+}1)$–RBC and let $A := (S_A, \Gamma_k, \delta_A, q_0, Q_{fin})$ be some blind k-counter automaton having $r + 1$ reversal bounded counters that accepts $B_{k,r} = L(A)$. Without loss of generality, we assume that the first $r + 1$ counters are reversal-bounded.

3.15 Definition

For each $k \in I\!N, k \neq 0$ and each $l, 1 \leq l < k$ let $\xi_k^l \subseteq G_A \times G_A$ be defined by:

$$(g, g') \in \xi_k^l \text{ iff } \begin{cases} \exists\, x \in \Gamma_k\text{:} & \pi_2(g), \pi_2(g') \in \{x, \lambda\}\ , \qquad \text{and} \\ \forall\, 1 \leq j \leq l\text{:} & \Delta_g(j) > 0 \Longrightarrow \Delta_{g'}(j) \geq 0, \\ & \Delta_{g'}(j) > 0 \Longrightarrow \Delta_g(j) \geq 0. \end{cases}$$

□

$(g, g') \in \xi_k^l$ means that the counter automaton A does not read two different symbols from the input by using g and g', if any at all, and these arcs do not force a reversal on any of the counters with index less or equal to l. The remaining counters with index strictly larger than l do not have any restriction on their updating. The relation ξ_k^l is obviously symmetric and reflexive but not necessarily transitive. So we can only find subsets $C \subseteq G_A \times G_A$ which are transitively closed. Any such set will be called a ξ_k^l-clique.

Within the set R_A we identify a certain non-regular subset K_1 to be used for the proof of Lemma 3.20 below.

3.16 Definition

The set

$$K_0 := \{w \in R_A \mid A_\Delta \cdot \psi(w) = 0 \text{ and } \exists\, i \in I\!N : \pi_2(w) = a_1^i b_1^i \cdots a_r^i b_r^i\, a_{r+1}^i b_{r+1}^{2i} a_{r+1}^i \cdots a_k^i b_k^{2i} a_k^i\}$$

is a non-regular subset of R_A of which we select the set $K_1 \subseteq K_0 \subsetneq R_A$ where no two different strings have an identical π_2-projection:

$$K_1 := \{w \in K_0 \mid \forall\, w' \in K_0 : \pi_2(w) = \pi_2(w') \text{ implies } w = w'\}.$$

By w_i we denote the unique string in K_1, for which

$$\pi_2(w_i) = a_1^i b_1^i a_2^i b_2^i \cdots a_r^i b_r^i a_{r+1}^i b_{r+1}^{2i} a_{r+1}^i \cdots a_k^i b_k^{2i} a_k^i.$$

□

For a step-by-step definition of a specific regular subset of R_A which contains infinitely many strings from the set K_1 we use the property $p_{\xi_k^{r+1}}$ to specify certain strings within the set K_1.

3.17 Definition

$p_{\xi_k^{r+1}} : G_A^* \times I\!N \to \{true, false\}$ is defined by:

$p_{\xi_k^{r+1}}(w,p) = true$ iff $\exists\, u_1, \ldots, u_p \in G_A^*$:

1. $w = u_1 u_2 \cdots u_p$ and
2. $\forall\, j, 1 \leq j \leq p : \forall\, g, g' \in G_A : g, g' \sqsubseteq u_j \Longrightarrow (g, g') \in \xi_k^{r+1}$
3. $\forall\, j, 1 \leq j < p : \exists\, g, g' \in G_A : g \sqsubseteq u_j \wedge g' \sqsubseteq u_{j+1} \wedge (g, g') \notin \xi_k^{r+1}$

For each $u \in G_A^*$ let $G(u) := \{g \in G_A \mid g \sqsubseteq u\}$, where $\sqsubseteq$ denotes the substring relation.

□

Here, $G(u_j)$ forms a ξ_k^{r+1}-clique for each u_j of the decomposition $w = u_1 u_2 \cdots u_p$. If two arcs $g, g' \in G_A$ are in the same ξ_k^{r+1}-clique, then there exists $x \in \Gamma_k$ such that $\pi_2(g), \pi_2(g') \in \{\lambda, x\}$ and their π_3-projections do not lead to a reversal on one of the first $r+1$ counters. The change between two ξ_k^{r+1}-cliques can thus be forced either by changing the symbols ($\neq \lambda$) of the π_2-projections or by performing a reversal on one of the first $r+1$ counters.

For each $w_i \in K_1$ we have $p_{\xi_k^{r+1}}(w_i, p) = true$ implies $p \leq 3k+1$. This is seen as follows: There exist exactly $2r + 3(k-r)$ different ξ_k^{r+1}-cliques with a component from Γ_k within the decomposition $w = u_1 u_2 \cdots u_p$, since there are that many different blocks of consecutive identical symbols. Because $(g, g') \in \xi_k^{r+1}$ also allows $\pi_2(g) = \pi_2(g') = \lambda$, some of these arcs may fall into the neighbouring ξ_k^{r+1}-clique, as long as these arcs do not force a reversal on one of the first $r+1$ counters. At most $r+1$ reversals may fall into the $2r + 3(k-r)$ different blocks, which allows for $r+1$ additional substrings in the decomposition of $w_i = u_1 u_2 \cdots u_p$ and $p \leq 2r + 3(k-r) + r + 1 = 3k+1$.

Since k is a constant and K_1 is infinite, there exists some $p \leq 3k+1$ such that infinitely many strings $w \in K_1$ satisfy $p_{\xi_k^{r+1}}(w,p) = true$. This gives rise to the subset $K_2 \subseteq K_1$ defined next.

3.18 Definition

Let $p \leq 3k+1$ be fixed and such that $K_2 := \{w \in K_1 \mid p_{\xi_k^{r+1}}(w,p) = true\}$ is infinite. Let $\#(K_2) := \{i \in I\!N \mid w_i \in K_2\}$ denote the index set for the strings in K_2. □

Since G_A is finite there exists a fixed string $w_g := g_{l,1} g_{l,2} \cdots g_{l,p} \in G_A^*$ where $g_{l,j}$ is the leftmost symbol of u_j for each $1 \leq j \leq p$ in the decomposition of $w = u_1 u_2 \cdots u_p$ for infinitely many strings $w \in K_2$. These strings are collected in the set $K_3 \subseteq K_2$:

3.19 Definition

Let $w_g = g_{l,1}g_{l,2}\cdots g_{l,p} \in G_A^*$ be fixed and such, that
$K_3 := K_2 \cap \{g_{l,1}\}G(u_1)^*\{g_{l,2}\}G(u_2)^*\cdots\{g_{l,p}\}G(u_p)^*$, is infinite. Moreover, $\#(K_3) := \{i \in I\!N \mid w_i \in K_3\}$ denotes the index set for the strings in K_3. $\square$

The set $K_3 \subseteq K_2 \subseteq K_1$ is not regular but we shall find an infinite regular set $L \subseteq \{g_{l,1}\}G(u_1)^*\{g_{l,2}\}G(u_2)^*\cdots\{g_{l,p}\}G(u_p)^*$ such that $K_3 \subsetneq L \subsetneq R_A$.

For each j, $1 \leq j \leq p$, let L_j be the regular set accepted by the finite Automaton $A_j := (Q_j, G(u_j), \delta_j, \pi_1(g_{l,j}), Q_{j,fin})$, where

1. $Q_j := \{\pi_1(g), \pi_4(g) \mid g \in G(u_j)\}$,
2. $\delta_j : Q_j \times G(u_j) \to Q_j$ is given by $\delta_j(\pi_1(g), g) := \pi_4(g)$,
3. $Q_{j,fin} := \pi_4(g')$, where g' is the rightmost symbol of u_j.

Since each accepting path in the automaton A_j is a part of an accepting path in $A's$ state diagram, we see that $K_3 \subseteq L \subseteq R_A$ for $L := L_1L_2\cdots L_p$. Moreover, at least $3k-r$ languages among the $L_1, L_2, \ldots .L_p$ must be infinite, since the projection of the elements of K_3 onto the elements of Γ_k are infinite for each of the $2r + 3(k-r)$ blocks of identical symbols. Since L is regular, the Parikh-image $\psi(L) = \sum_{1\leq j\leq p}\psi(L_j)$ is a semilinear set and infinite, too. The sum is understood elementwise for the p semilinear sets $\psi(L_j)$. Each linear subset of $\psi(L_j)$ has a representation of the form:

$$\{C_j + P_jY \mid Y \in I\!N^{h_j}\} \text{ for some } h_j \geq 1,\ C_j \in I\!N^{n_A}, \text{ and } P_j \in I\!N^{n_a \times h_j}.$$

With these preliminaries we can formulate and prove the following important result:

3.20 Lemma

There exists an infinite set $K \subseteq R_A$ such that a) to c) hold:

a) $\psi(K) = \{C + PY \mid Y \in I\!N^h\}$ for some $h \in I\!N$, $C \in I\!N^{n_A}$, and $P \in I\!N^{n_A \times h}$,

b) If $P(s,j)\cdot P(t,j) \neq 0$ for $1 \leq j \leq h$, $1 \leq s,t \leq n_A$ then $(g_s, g_t) \in \xi_k^{r+1}$,

c) $\forall\, n_0 \in I\!N : \exists\, Y_0 \in I\!N^h : (\forall\, j : 1 \leq j \leq h \wedge Y_0(j) > n_0) \wedge C + PY_0 \in \psi(K_1)$.

Proof: By definition of the finite automata A_j, $1 \leq j \leq p$, each matrix P_j satisfies b) of Lemma 3.20. Given $L := L_1L_2\cdots L_p$ we choose for each L_j a linear subset $\{C_j + P_jY \mid Y \in I\!N^{h_j}\} \subseteq \psi(L_j)$ which should be infinite whenever L_j is infinite. The set $S := \{C_S + P_SY \mid Y \in I\!N^{h_S}\}$ defined by $C_S := \sum_{j=1}^{p} C_j$, $h_S := \sum_{j=1}^{p} h_j$, and the compound matrix $P_S :=$

$\left(P_1\, P_2\, \cdots\, P_p\right) \in I\!N^{n_A \times h_S}$ is linear and infinite, too. The matrix P_S satisfies property b) of Lemma 3.20, since each submatrix P_j fulfilled this property. Now, $L' := L \cap \psi^{-1}(S)$ is an infinite subset of L containing infinitely many elements from $K_3 \subseteq K_1$, thus satisfying properties a) (by $\psi(L') = S$) and b) of Lemma 3.20. Of course we had to choose the appropriate linear subsets of each L_j to see that $\psi^{-1}(S)$ contains infinitely many elements of $K_3 \subsetneq L = L_1 L_2 \cdots L_p$. Now we modify the matrix P_S by omitting certain columns to obtain a matrix P that also satisfies c) of the lemma. First, $L' \cap K_3 \subseteq R_A$ is infinite, so that there exists an infinite set $M \subseteq I\!N^{h_S}$ such that $\psi(L' \cap K_3) = \{C_S + P_S Y \mid Y \in M\}$. Let $m_0, m_1, \ldots, m_i, \ldots$ be any fixed enumeration of the elements of $M = \{m_i \mid i \in I\!N\}$. Then there exists a subset $M' = \{m_{i_j} \mid \forall\, j \in I\!N : i_j \in I\!N \wedge m_{i_j} \in M \wedge i_j < i_{j+1}\} \subseteq M$ such that for each j, $1 \leq j \leq h_S$:

either $m_{i_1}(j) = m_{i_2}(j)$ for all $i_1, i_2 \in I\!N$,

or $m_{i_1}(j) < m_{i_2}(j)$ for all $i_1 < i_2$.

This result is a variant of Dickson's Lemma and is shown in the appendix as Lemma 4.3. From M' we deduce the following index sets and constants:

$I_{le} := \{j \mid 1 \leq j \leq h_S, \forall\, l \geq 1 : m_{i_l}(j) < m_{i_{l+1}}(j)\}$,

$I_{eq} := \{j \mid 1 \leq j \leq h_S, \forall\, l \geq 1 : m_{i_l}(j) = m_{i_{l+1}}(j)\}$, and

$c_j := m_{i_1}(j)$ for each $j \in I_{eq}$.

Now,
$\psi(L' \cap K_3) = \{C_S + P_S Y \mid Y \in M\} \supseteq \{C_S + \sum_{j \in I_{eq}} P_S(:,j) c_j + PY \mid Y \in M''\}$, where $P \in I\!N^{n_A \times h}$ is obtained from P_S by omitting the columns $P_S(:,j)$ having index $j \in I_{eq}$, $h := h_S - |I_{eq}|$, and M'' is obtained from M' by omitting all components j, where $j \in I_{eq}$. Thereby, $M'' \subsetneq I\!N^h$ is a set which can be linearly ordered by $<$ and this relation applies to all components of its elements. We now can identify the set K from Lemma 3.20 as $K := L' \cap K_3 \cap \psi^{-1}(\{C_S + \sum_{j \in I_{eq}} P_S(:,j) c_j + PY \mid Y \in M''\})$ Thus, all properties a), b) and c) of Lemma 3.20 are satisfied, and the proof is finished. □

Let P be the matrix defined in Lemma 3.20 above, then we have:

3.21 Lemma

$$rank\left(\begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot P\right) \quad > \quad rank\,(A_\Delta \cdot P)\,.$$

Proof: Let $\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \\ Z_1 & Z_2 & \cdots & Z_h \end{pmatrix} = \begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot P$, where for each $1 \leq j \leq h$, the columns $Y_j := (\,A_\Delta \cdot P\,)\,(:,j)$ of $A_\Delta \cdot P$ are given by $Y_j(l) := \sum_{i=1}^{n_A} A_\Delta(l,i) P(i,j) := \sum_{i=1}^{n_A} \Delta_{g_i}(l) \cdot P(i,j)$ for $1 \leq l \leq k$ and likewise $Z_j := (\,A_\Gamma \cdot P\,)\,(:,j)$ denotes the j-th column of $A_\Gamma \cdot P$ with $Z_j(l) := \sum_{i=1}^{n_A} A_\Gamma(l,i) P(i,j)$. From b) in Lemma 3.20 one concludes that each column Z_j,

$1 \le j \le h$, has at most one non-zero component: if $Z_j(l) \neq 0$ then $Z_j(i) = 0$ for each $i \neq l$. (A detailed exposition of this fact can be found in the appendix as Lemma 4.2.)

We still have to verify: $rank \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \\ Z_1 & Z_2 & \cdots & Z_h \end{pmatrix} > rank \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \end{pmatrix}$. By the definition of matrix A_Γ (Def. 3.10) and the construction of P (Lemma 3.20) one readily verifies that the rows of the compound matrix $\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$ are linearly independent. For later use, let $\alpha_1, \alpha_2, \dots \alpha_k$ and $\beta_1, \beta_2, \dots \beta_k$ denote the rows of $\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \end{pmatrix}$, respectively those of $\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$.

Each word $w_i \in K \cap K_3$, $i \in \#(K_3)$, can be written as
$w_i = u_{1,1}^{(i)} u_{1,2}^{(i)} u_{2,1}^{(i)} u_{2,2}^{(i)} \cdots u_{r,1}^{(i)} u_{r,2}^{(i)} w_{r+1,1}^{(i)} w_{r+1,2}^{(i)} w_{r+1,3}^{(i)} \cdots w_{k,1}^{(i)} w_{k,2}^{(i)} w_{k,3}^{(i)}$, where

1. $\pi_2(u_{s,1}^{(i)}) = a_s^i$ and $\pi_2(u_{s,2}^{(i)}) = b_s^i$ for each s, $1 \le s \le r$,
2. $\pi_2(w_{s,1}^{(i)}) = \pi_2(w_{s,3}^{(i)}) = a_s^i$ and $\pi_2(w_{s,2}^{(i)}) = b_s^{2i}$ for each s, $r+1 \le s \le k$.

If $P(i,j) \neq 0$ then for each $n_0 \in I\!N$ there exists $w \in K$ such that $|w|_{g_i} > n_0$. This follows from c) in Lemma 3.20. Let $G_1 := \{g \in G_A \mid \forall n_0 \in I\!N : \exists\, w \in K : |w|_{g_i} > n_0\}$ be the set of all these arcs. Now we want to show that the row-space $\{\alpha_1, \alpha_2, \dots \alpha_k, \beta_1, \beta_2, \dots \beta_k\}$ contains strictly more linearly independent elements than the row-space $\{\beta_1, \beta_2, \dots \beta_k\}$ if $B_{k,r} = L(A)$ for the counter automaton A having r+1 reversal bounded counters. We distinguish two cases:

1. Assume that one of the reversal bounded counters will be changed by an arc $g \in G(w_{l,1}^{(i)} w_{l,2}^{(i)} w_{l,3}^{(i)}) \cap G_1$ for some l, $r+1 \le l \le k$. W.l.o.g. we assume that this is the first counter, hence $\Delta_g(1) = \pi_3(g)(1) \neq 0$. By choosing two more arcs from $G(w_{l,1}^{(i)} w_{l,2}^{(i)} w_{l,3}^{(i)}) \cap G_1$ we can always find three elements $g_{\mu_1}, g_{\mu_2}, g_{\mu_3} \in G_A$, $1 \le \mu_1, \mu_2, \mu_3 \le n_A$, such that:

1. $g \in \{g_{\mu_1}, g_{\mu_2}, g_{\mu_3}\}$,
2. $g_{\mu_1} \sqsubseteq w_{l,1}^{(i)}$, $g_{\mu_2} \sqsubseteq w_{l,2}^{(i)}$, and $g_{\mu_3} \sqsubseteq w_{l,3}^{(i)}$,
3. if $g \neq g_{\mu_j}$ for some $1 \le j \le 3$ then $\pi_2(g_{\mu_j}) \neq \lambda$.

Now consider two triples $\vec{y} := (y_1, y_2, y_3)$ and $\vec{z} := (z_1, z_2, z_3)$, where y_1, y_2, and y_3 are entries of the matrix $\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \end{pmatrix}$ and z_1, z_2, and z_3 are entries of the matrix $\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$. $\vec{y}$ and $\vec{z}$ are specified as follows: for $1 \le j \le 3$ the elements y_j are located in the first row α_1 of $\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \end{pmatrix}$ and the elements z_j are located in the l-th row β_l of $\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$ and their crossing with some column $\begin{pmatrix} Y_{m_j} \\ Z_{m_j} \end{pmatrix}$, where $1 \le m_j \le h$ for $1 \le j \le 3$, and m_j is such, that $P(\mu_j, m_j) \neq 0$. As mentioned before, each column of $\begin{pmatrix} Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$ has at most one single entry not equal to *zero*. Since $\pi_2(g_{\mu_1}), \pi_2(g_{\mu_2}), \pi_2(g_{\mu_3}) \in \{a_l, b_l, \lambda\}$ these entries must occur in the l-th row β_l of $\begin{pmatrix} Z_{m_1} & Z_{m_2} & Z_{m_3} \end{pmatrix}$, hence applies to the elements z_1, z_2, and z_3. Consequently, if $\vec{y}$ was linearly independent of $\vec{z}$ then also the first row α_1 would be linearly independent of the rows $\beta_1, \beta_2, \ldots, \beta_k$. This would imply the statement of Lemma 3.21. Thus it suffices to prove that indeed $\vec{y}$ and $\vec{z}$ are linearly independent.

Among the cases $g = g_{\mu_1}$, $g = g_{\mu_2}$, or $g = g_{\mu_3}$ we select $g := g_{\mu_1}$ as subcase **1.1** (the remaining cases are similar):

By the choice of g we have $\Delta_g(1) \neq 0$ which implies $y_1 \neq 0$. Now, either $\vec{z} = (0, -j_z, +k_z)$ or $\vec{z} = (+i_z, -j_z, +k_z)$ for some $i_z, j_z, k_z \geq 1$ by definition of $\{g_{\mu_1}, g_{\mu_2}, g_{\mu_3}\}$. Since $\vec{z} = (0, -j_z, +k_z)$ means independence of $\{\vec{y}, \vec{z}\}$ we proceed by assuming $\vec{z} = (+i_z, -j_z, +k_z)$. Since the first counter is reversal bounded, only the following choices are possible for $\vec{y}$: $y_1 > 0, y_2 > 0, y_3 \neq 0$, $y_1 > 0, y_2 < 0, y_3 < 0$, or $y_1 < 0, y_2 < 0, y_3 < 0$. It is immediately verified that in all these cases $\vec{y}$ is linearly independent of $\vec{z}$.

2. We next have to consider the case, that for each $l, r+1 \leq l \leq k$ no arc $g \in G(w^{(i)}_{l,1} w^{(i)}_{l,2} w^{(i)}_{l,3}) \cap G_1$ updates one of the r+1 reversal bounded counters. Let $G_2 := G(w^{(i)}_{r+1,1} w^{(i)}_{r+1,2} w^{(i)}_{r+1,3} \cdots w^{(i)}_{k,1} w^{(i)}_{k,2} w^{(i)}_{k,3}) \cap G_1$ be the relevant set of these arcs. Again we consider matrices defined from columns of $\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_h \\ Z_1 & Z_2 & \cdots & Z_h \end{pmatrix}$ as follows:

Let $(Y_{m_1} \quad Y_{m_2} \quad \cdots \quad Y_{m_q})$ and $(Z_{m_1} \quad Z_{m_2} \quad \cdots \quad Z_{m_q})$ be the matrices consisting of those columns of $(Y_1 \quad \cdots \quad Y_h)$, respectively of $(Z_1 \quad \cdots \quad Z_h)$, for which $P(j, m_i) \neq 0$ and $g_j \in G_2$ where $1 \leq j \leq n_A$, and $1 \leq m_i \leq h$ for all $1 \leq i \leq q$. Now, $g \in G_2$ implies $\Delta_g(j) = 0$ for $1 \leq j \leq r+1$, since none of the reversal bounded counters is modified by an arc from the set G_2. Consequently $Y_{m_i}(j) = 0$ for $1 \leq j \leq r+1$ and each $1 \leq i \leq q$, so that $\mathit{rank}\,(Y_{m_1} \quad Y_{m_2} \quad \cdots \quad Y_{m_q}) \leq k - (r+1)$. On the other hand, $\mathit{rank}\,(Z_{m_1} \quad Z_{m_2} \quad \cdots \quad Z_{m_q}) = k - r$, since $\mathit{rank}\,(Z_1 \quad Z_2 \quad \cdots \quad Z_h) = k$ and r rows of $(Z_{m_1} \quad Z_{m_2} \quad \cdots \quad Z_{m_q})$ have an entry equal to *zero* (recall $\pi_2(g) \notin \Gamma_r$ for $g \in G_2$). Also in case **2.** the statement of the lemma has been proved. □

Proof of Lemma 3.14: From Lemma 3.12 we see that $L(A) = B_{k,r} \subseteq C_k$ implies $\{v \in R_A \mid A_\Delta \cdot \psi(v) = \vec{0}\} = \{v \in R_A \mid \begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot \psi(v) = \vec{0}\}$. Using $K \subseteq R_A$ from Lemma 3.20 a) with $\psi(K) = \{C + P \cdot Y \mid Y \in I\!N^h\}$ and by b) there exists $Y_0 \in I\!N^h$ for each $n_0 \in I\!N$ with $Y_0(j) > n_0$ for each $1 \leq j \leq h$ and a string $w \in R_A$ such that $\pi_2(w) \in B_{k,r}$, $A_\Gamma \cdot \psi(w) = 0$, $A_\Delta \cdot \psi(w) = 0$, and $C + P \cdot Y_0 = \psi(w)$. This yields the equation

$$(*): \quad \left\{ Y \in I\!N^h \,|\, A_\Delta \cdot P \cdot Y = -(A_\Delta) \cdot C \right\} = \left\{ Y \in I\!N^h \,\middle|\, \begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot P \cdot Y = -\begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot C \right\}, \text{ and}$$

$Y_0 \in \left\{ Y \in I\!N^h \,|\, A_\Delta \cdot P \cdot Y = -(A_\Delta) \cdot C \right\}$. But by Lemma 4.1 and Lemma 3.21 we see

$$\mathit{rank} \left\{ Y \in I\!N^h \,|\, A_\Delta \cdot P \cdot Y = -(A_\Delta) \cdot C \right\} > \mathit{rank} \left\{ Y \in I\!N^h \,\middle|\, \begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot P \cdot Y = -\begin{pmatrix} A_\Delta \\ A_\Gamma \end{pmatrix} \cdot C \right\},$$

which means that equation $(*)$ cannot be fulfilled and $B_{k,r} \notin (k, r+1)\text{–}RBC$. $\square$

The above results yield our main Theorem:

3.22 Theorem

$(k_1, r_1)\text{–}RBC \subsetneq (k_2, r_2)\text{–}RBC$ iff $(k_1 < k_2)$ or $(k_1 = k_2$ and $r_1 > r_2)$.

Proof: The mere inclusion $(k_1, r_1)\text{–}RBC \subseteq (k_2, r_2)\text{–}RBC$ if $(k_1 < k_2)$ or $(k_1 = k_2$ and $r_1 > r_2)$, follows from the definition of the family $(k, r)\text{–}RBC$ (Definitions 3.1 to 3.3). The strictness of $(k_1, r_1)\text{–}RBC \subsetneq (k_2, r_2)\text{–}RBC$ if $k_1 < k_2$ is verified as follows:

By definition we have $(k, r_1)\text{–}RBC \subseteq (k, 0)\text{–}RBC$ for any $r_1 \leq k$, the strict inclusion $(k, 0)\text{–}RBC = \mathcal{M}(C_k) \subsetneq \mathcal{M}(B_{k+1}) = (k+1, k+1)\text{–}RBC$ is Theorem 3.6, and again by definition $(k+1, k+1)\text{–}RBC \subseteq (k+1, r_2)\text{–}RBC$ for any $r_2 \leq k+1$. Finally, the inclusion $(k, r+1)\text{–}RBC \neq (k, r)\text{–}RBC$ for all $k \in I\!N$ and $0 \leq r < k$ has been shown in Lemma 3.14. $\square$

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4 Appendix

Let $A \in \mathbb{Z}^{n \times h}$ be of rank r, $B \in \mathbb{Z}^h$, and $L := \{x \in \mathbb{N}^h \mid Ax = B\}$ be the set of all non-negative solutions of the linear equation $Ax = B$. It is known from Linear Algebra that each subset $M \subseteq L$ of linearly independent elements has cardinality of at most $(h-r)$.

4.1 Lemma

If $L := \{x \in \mathbb{N}^h \mid Ax = B\}$ for some $A \in \mathbb{Z}^{n \times h}$ of rank r and $B \in \mathbb{Z}^h$. If for each $n \in \mathbb{N}$ there exists $x \in L$ such that $x(i) > n$ for each i, $1 \leq i \leq h$, then L contains a subset $M = \{x_1, x_2, \ldots, x_{h-r}\}$ of linearly independent elements.

Proof: Let $y_1, y_2, \ldots, y_{h-r} \in \mathbb{Z}^h$ be linearly independent solutions of the homogenous linear equation $Ax = 0$ and define $n_0 := max\{\,|y_i(j)|\,|1 \leq i \leq h, 1 \leq j \leq r-h\}$. Now, if $x_0 \in L$ is a solution of the inhomogeneous linear equation $Ax = B$ that satisfies $x_0(i) > n_0$, then $x_0 + y_1$, $x_0 + y_2$,$\ldots x_0 + y_{h-r}$ are linearly independent and non-negative solutions of the equation $Ax = B$. □

4.2 Lemma

If $(\,Z_1 \quad Z_2 \quad \cdots \quad Z_h\,) = A_\Gamma \cdot P$ then $Z_j(l) \neq 0$ implies $Z_j(i) = 0$ for each $i \neq l$.

Proof: $A_\Gamma(i,s) \cdot P(s,j) \neq 0$ and $A_\Gamma(l,t) \cdot P(t,j) \neq 0$ implies $P(s,j) \cdot P(t,j) \neq 0$, hence $(g_s, g_t) \in \xi_k^{r+1}$ by b) of Lemma 3.20. This in turn means that there exists an $x \in \Gamma_k$ such that $\pi_2(g_s) \in \{x, \lambda\}$ and $\pi_2(g_t) \in \{x, \lambda\}$ and $i = l$ follows from the structure of matrix A_Γ. Furthermore, $A_\Gamma(i,s) = A_\Gamma(i,t) \in \{1, -1\}$ and the sum $Z_j(l) := \sum_{i=1}^{n_A} A_\Gamma(l,i)P(i,j)$ will never be *zero*. □

4.3 Lemma

Each infinite sequence $\sigma := m_1, m_2, \ldots, m_i, \ldots$ of elements $m_i \in N^n$ has an infinite subsequence $\sigma' := m_{i_1}, m_{i_2}, \ldots, m_{i_r}, \ldots$, such that $r < s$ implies $i_r < i_s$ and for each $j, 1 \leq j \leq n$ the following holds: either $m_{i_r}(j) < m_{i_{r+1}}(j)$ for all $r \geq 1$ or $m_{i_r}(j) = m_{i_{r+1}}(j)$ for all $r \geq 1$.

This Lemma is similar to the one by Dickson, stating that each infinite sequence of vectors contains an infinite nondecreasing subsequence, and only slightly stronger than this.

Proof: For each infinite sequence $\sigma := m_1, m_2, \ldots, m_i, \ldots$ of elements $m_i \in N^n$ let $M_\sigma := \{m_i \mid m_i \text{ occurs within } \sigma\}$. We proceed by induction on n:

Basic step: For $n = 1$, $\sigma := m_1, m_2, \ldots, m_i, \ldots$ is an infinite sequence of not necessarily different elements. If M_σ is finite, then there exists (at least one) $m \in M_\sigma$ such that $\{i \mid m_i = m\}$ is infinite and the following subsequence σ' can be chosen: $\sigma' := m_{i_1}, m_{i_2}, \ldots, m_{i_j}, \ldots$ where $m_{i_r} = m$ for each $r \geq 1$. Here $\{j \mid m = m_j\} = \{i_j \mid m = m_{i_j}\}$ and σ' satisfies the requirement.

If, on the other hand, M_σ is not finite, then there exists an infinite subsequence of the form $(\star)$.

$(\star)$: M_σ is infinite and each $m \in M_\sigma$ occurs exactly once within the sequence σ', that is, $m_{i_s} \neq m_{i_t}$ iff $s \neq t$.

This sequence will be defined by repeatedly applying the following transformation **R** on the current sequence σ. **R** has two parameters: σ and some $i \in I\!N$ and is defined as follows:

$\mathbf{R}(\sigma, i)$ means: remove all elements m_t from σ for which $t > i$ and $m_t = m_i$.

If $\sigma' :=\mathbf{R}(\sigma, i)$ is the infinite sequence obtained by performing **R** once for some fixed $i \in I\!N$, then obviously $M_\sigma = M_{\sigma'}$ and σ' still is an infinite sequence. Hence, if σ' is the sequence obtained from σ successively applying $\mathbf{R}(\sigma, i)$ for each $i \geq 1$, then σ' has the desired property $(\star)$.

Starting with an infinite sequence $\sigma := m_1, m_2, \ldots, m_i, \ldots$, $(m_i \in I\!N)$, that satisfies $(\star)$, the Lemma of Dickson shows, that there exists an infinite subsequence $\sigma' := m_{i_1}, m_{i_2}, \ldots, m_{i_j}, \ldots$ of σ where $m_{i_r} < m_{i_{r+1}}$ for each $r > 1$. and this proves the induction basis.

Induction step: Assuming the Lemma to be true for some fixed $m \in I\!N$ let $\sigma := m_1, m_2, \ldots, m_i, \ldots$ be an infinite sequence of elements $m_i \in N^{m+1}$. By considering the first m components of the elements of σ there exists an infinite subsequence $\sigma' := m_{i_1}, m_{i_2}, \ldots, m_{i_j}, \ldots$ of σ such that the projection onto the first m components yields an infinite subsequence for which the lemma holds. Now consider the last component of the elements forming the sequence σ'. Apply the selection mechanism to the elements of σ' according to the basic step applied to the one dimensional sequence in the $(m + 1)$-st coordinate of the elements in σ' finally yields an infinite subsequence of σ', hence of σ, which satisfies the requirements of the lemma in all coordinates. □